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# Grassmann structures and integrable systems

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## Abstract

We consider some natural connections which arise between right-flat  $(p, q)$  paraconformal structures and integrable systems. We find that such systems may be formulated in Lax form with a “Lax  $p$ -tuple” of linear differential operators, depending a spectral parameter which lives in  $(q - 1)$ -dimensional complex projective space. Generally, the differential operators contain partial derivatives with respect to the spectral parameter. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

It has long been known that in four-dimensional Riemannian geometry there is a connection between conformal structures with anti-self-dual Weyl tensor, and three-dimensional complex manifolds:

**Theorem 1.1** (Atiyah, Hitchin and Singer [1]). *If a manifold  $M$  admits a conformal structure with  $W^+ = 0$ , then the projective spin-bundle  $\mathbf{P}(\mathbb{V}^+)$  is a complex 3-manifold. Conversely, given a complex 3-manifold  $Z$  with a real structure (i.e. anti-holomorphic involution)  $\sigma$  and a 4 parameter family of embedded rational curves with normal bundle  $N \cong H \oplus H$ , on which  $\sigma$  acts as the anti-podal map, then the space of real rational curves admits an anti-self-dual conformal structure. All anti-self-dual conformal structures arise in this way.*

Moreover, if one considers anti-self-dual Yang–Mills fields on an anti-self-dual background, then solutions may be constructed in terms of holomorphic vector bundles on this

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complex 3-manifold. The existence of such a complex 3-manifold is looked on as being central to the notion of integrability of the anti-self-duality equations [9,16,17].

If one tries to generalise notions of anti-self-duality to higher dimensional Riemannian manifolds, there are several (inequivalent) paths one may choose to follow.

A common choice is to investigate Riemannian manifolds in higher dimension with reduced holonomy group [15], and gauge fields related to these structures [4]. In the case of (irreducible, non-symmetric) Riemannian manifolds, one may then invoke Berger's classification of the possible holonomy groups of the Levi-Civita connection. Apart from the generic case of  $SO(n)$  holonomy group, the holonomy groups allowed by Berger's classification correspond to Kähler, quaternionic-Kähler, Ricci-flat Kähler and hyper-Kähler manifolds, along with two exceptional possibilities of Ricci-flat metrics with holonomy group  $G_2$  and  $Spin_7$  in dimensions 7 and 8. This may not be the most natural approach if ones interest is in integrable systems, however, since, of these possibilities, the only systems which appear to be integrable are those which govern Kähler, quaternionic-Kähler and hyper-Kähler structures. The equations for Ricci-flat Kähler metrics are not integrable in dimensions greater than 4. Little is known concerning the integrability of the  $G_2$  and  $Spin_7$  holonomy equations, although they contain as special cases the (non-integrable) equations for six-dimensional and eight-dimensional Ricci-flat Kähler structures, respectively. Therefore, it is extremely unlikely that these two systems are integrable.

An alternative path is not to start with the geometrical condition of anti-self-duality of the four-dimensional metric, but to simply consider systems in higher dimensions where there is a suitable generalisation of the complex 3-manifold which appears in four dimensions. Generically, we will denote such a complex manifold by  $Z$ , and the idea is that one reconstructs the geometrical manifold, denoted  $M$ , as a parameter space of particular sub-manifolds of  $Z$ . Natural geometrical structures then arise on the manifold  $M$  as a result of the integrability of the complex structure on  $Z$ . Any additional holomorphic structures that exist on  $Z$  then lead to more specialised geometrical structures on  $M$ . The complex manifold  $Z$  will be referred to as a twistor space, and the essence of the work of Ward and others [9,16,17] is that it is the existence of a twistor construction for a problem which should be interpreted as a sign of its integrability. Substantial evidence for this claim comes from the fact that many standard integrable systems in two and three dimensions may be constructed as symmetry reductions of the equations for anti-self-dual Yang-Mills fields and anti-self-dual conformal structures in four dimensions [9,17]. In this approach it is the existence of the complex manifold  $Z$  that is central, the equations on the space-time then being simply a manifestation of the complex structure on  $Z$ .

We aim here to study (local) properties of structures for which there is a complex manifold construction, the right-flat Grassmann structures, and to see what new features of integrable systems these structures suggest. In particular, we begin by reviewing, in explicitly local terms, the construction implicit in Theorem 1.1. What we find is that even in this simplest situation, there are features of the equations which arise which are unusual from the point of view of integrable systems. In the generic case of an anti-self-dual manifold, the spin-bundle does not fibre over  $\mathbf{P}_1$ , the complex projective line. In integrable systems terms this means that the operators in our Lax pair contain partial derivatives with respect to the spectral parameter. Therefore the spectral parameter itself is very much part of the geometrical problem, a property which is unusual (but not unknown) in conventional integrable systems theory.

We then study, in a similar fashion  $(p, q)$  right-flat Grassmann structures in dimension  $n = pq$ . In this case, we find that as opposed to a Lax pair of operators depending on a spectral parameter  $\lambda \in \mathbf{P}_1$ , the right-flat condition on a Grassmann structure is determined by a Lax  $p$ -tuple of differential operators depending on a spectral parameter taking values in the higher-dimensional projective space  $\mathbf{P}_{q-1}$ . As in the case of anti-self-dual structures in four dimensions, these differential operators generally contain derivatives in the spectral parameter, corresponding to the fact that the complex manifold  $Z$  does not fibre holomorphically over  $\mathbf{P}_{q-1}$ .

The moral of our story is that if one takes the ideas of Ward and others seriously that it is the connection with complex manifold theory which is central to integrable system theory, then one must substantially generalise what one considers to be an integrable system.

## 2. Anti-self-dual conformal structures

We begin by reviewing, in local terms, the construction implicit in Theorem 1.1. Consider an oriented Riemannian four-manifold  $M$ . We may then define a canonical almost complex structure on the projective spin-bundle. First, we use the Levi–Civita connection to split the tangent bundle of  $\mathbf{P}(\mathbb{V}^+)$  as the direct sum of a vertical part along the fibres,  $V(\mathbf{P}(\mathbb{V}^+))$ , and the horizontal part,  $H(\mathbf{P}(\mathbb{V}^+))$ , which is the pull-back of the tangent bundle of  $M$ ,  $p^*TM$ . The vertical fibres are complex projective lines, and so inherit a natural almost complex structure. In the horizontal direction, a non-zero spinor  $\pi \in (\mathbb{V}^+)_x$  identifies  $T_xM$  by Clifford multiplication with the two-dimensional complex vector space  $(\mathbb{V}^-)_x$ . At the points of  $\mathbf{P}(\mathbb{V}^+)$  corresponding to  $\pi$ , we put this almost complex structure on  $H_x(\mathbf{P}(\mathbb{V}^+))$ . It follows that this almost complex structure is integrable if and only if the Weyl tensor of the conformal structure is anti-self-dual [1].

To cast this in more explicit terms, fix a Riemannian metric,  $\mathbf{g}$ , in the conformal structure. If we complexify the tangent space, and extend the metric by complex linearity to a complex metric (again denoted  $\mathbf{g}$ ) on  $TM \otimes \mathbb{C}$ , then locally, we may introduce a null basis  $\{\epsilon^i | i = 1, \dots, 4\}$  for  $T^*M \otimes \mathbb{C}$  in which the metric may be written as

$$\mathbf{g} = \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 + \epsilon^3 \otimes \epsilon^4 + \epsilon^4 \otimes \epsilon^3. \tag{2.1}$$

We can then define the Levi–Civita connection,  $\Gamma$ , of the tetrad by the equation

$$d\epsilon^i + \sum_{j=1}^4 \Gamma^i_j \wedge \epsilon^j = 0, \quad i = 1, \dots, 4.$$

If we adopt an affine complex coordinate,  $\lambda$ , on the fibre  $(\mathbf{P}(\mathbb{V}^+))_x \cong \mathbf{P}_1$ , then we define an almost complex structure on  $\mathbf{P}(\mathbb{V}^+)$  by defining the distribution  $\Lambda \subset T^*(\mathbf{P}(\mathbb{V}^+))$  spanned by the 1-forms

$$\sigma_1 = \epsilon^3 + \lambda \epsilon^1, \quad \sigma_2 = \epsilon^2 - \lambda \epsilon^4, \quad \sigma_3 = d\lambda + \Gamma_{14} + \lambda(\Gamma_{12} - \Gamma_{34}) + \lambda^2 \Gamma_{23}.$$

This almost complex structure on  $\mathbf{P}(\mathbb{V}^+)$  is integrable if and only if the distribution  $\Lambda$  is involutive, i.e.  $d\Lambda \subset \Lambda^1 \wedge \Lambda$ . It is straightforward to show that this is the case if and

only if the Weyl tensor of the metric  $\mathbf{g}$  defined above is anti-self-dual. It also follows straightforwardly that this construction is unaffected by conformal changes of metric, and so depends only on the conformal equivalence class of the metric [1].

The connection with integrable systems comes from taking a dual formulation of this result [9]. The anti-holomorphic tangent space of  $\mathbf{P}(\mathbb{V}^+)$  is spanned by the vector fields

$$\begin{aligned} \mathbf{v}_1 &= \frac{1}{1 + \lambda\bar{\lambda}} \left[ \mathbf{e}_4 + A_4 \frac{\partial}{\partial \lambda} + \bar{A}_4 \frac{\partial}{\partial \bar{\lambda}} + \lambda \left( \mathbf{e}_2 + A_2 \frac{\partial}{\partial \lambda} + \bar{A}_2 \frac{\partial}{\partial \bar{\lambda}} \right) \right], \\ \mathbf{v}_2 &= \frac{1}{1 + \lambda\bar{\lambda}} \left[ \mathbf{e}_1 + A_1 \frac{\partial}{\partial \lambda} + \bar{A}_1 \frac{\partial}{\partial \bar{\lambda}} - \lambda \left( \mathbf{e}_3 + A_3 \frac{\partial}{\partial \lambda} + \bar{A}_3 \frac{\partial}{\partial \bar{\lambda}} \right) \right], \\ \mathbf{v}_3 &= \frac{\partial}{\partial \bar{\lambda}}, \end{aligned}$$

where  $\mathbf{e}_i$  are vector fields on  $M$  dual to the 1-forms  $\epsilon^i$

$$\langle \epsilon^i, \mathbf{e}_j \rangle = \delta^i_j,$$

and

$$\mathbf{A} = -\Gamma_{14} - \lambda(\Gamma_{12} - \Gamma_{34}) - \lambda^2\Gamma_{23}, \quad \bar{\mathbf{A}} = -\Gamma_{23} + \lambda(\Gamma_{12} - \Gamma_{34}) - \lambda^2\Gamma_{14}.$$

The complex structure defined by these vectors is integrable if they are closed under Lie brackets. We now note that the complex structure defined by these vector fields is the same as defined by the following basis:

$$\mathbf{L}_1 = D_4 + \lambda D_2, \quad \mathbf{L}_2 = D_1 - \lambda D_3, \quad \mathbf{v} = \frac{\partial}{\partial \bar{\lambda}},$$

where we have defined the vector fields

$$D_i = \mathbf{e}_i + A_i \frac{\partial}{\partial \lambda}. \tag{2.2}$$

The only non-trivial part of the integrability of the complex structure we have defined is that the Lie bracket of  $\mathbf{L}_1$  and  $\mathbf{L}_2$  must lie in  $T^{(0,1)}$ . Therefore for integrability we require the existence of functions  $\alpha(x : \lambda), \beta(x : \lambda)$  with the property that

$$[D_4 + \lambda D_2, D_1 - \lambda D_3] = \alpha(D_4 + \lambda D_2) + \beta(D_1 - \lambda D_3). \tag{2.3}$$

A power counting argument implies that the functions  $\alpha$  and  $\beta$  are quadratic polynomials in the variable  $\lambda$ . If this condition is satisfied, then the projective spin-bundle is a complex 3-manifold, and so the conformal structure must be anti-self-dual. Conversely, if the conformal structure is anti-self-dual, then the projective spin-bundle is a complex 3-manifold and so, locally, we may choose bases where the above equations are satisfied. We therefore have the following theorem.

**Theorem 2.1.** *Given an anti-self-dual conformal structure and any representative metric in the conformal class written in the form (2.1), then there exists a 1-form  $\mathbf{A}$ , which is a quadratic function of an arbitrary  $\mathbf{P}_1$ -valued parameter  $\lambda$  and two quadratic functions of  $\lambda, \alpha$  and  $\beta$  which obey Eq. (2.3), where the differential operators  $D_i$  are as in Eq. (2.2).*

It is possible to decompose Eq. (2.3) into components in the tangent space of the manifold  $M$  and components in the vertical direction  $\partial/\partial\lambda$ . The components in  $TM$  tell us that the functions  $\alpha, \beta$  and the components of the form  $\mathbf{A}$  correspond to parts of the Levi–Civita connection. The parts of the Levi–Civita connection they define are precisely the parts required to construct the self-dual part of the Weyl tensor,  ${}^+W$ . The vertical component of Eq. (2.3) then tell us that the five individual components of  ${}^+W$  vanish identically, so the Weyl tensor is anti-self-dual.

Eq. (2.3) tells us that the operators  $\mathbf{L}_1$  and  $\mathbf{L}_2$  constitute a Lax pair for the problem, and therefore that the system is integrable. However, these operators contain derivatives with respect to the spectral parameter  $\lambda$ , a feature which does not usually occur in standard integrable systems theory. The origin of these derivative terms lies in the nature of the complex manifold  $\mathbf{P}(\mathbb{V}^+)$ . Eq. (2.3) are the integrability condition which ensures the existence of three linearly independent solutions of the over-determined set of equations for a function  $f(x : \lambda)$

$$\mathbf{L}_1 f = \mathbf{L}_2 f = 0.$$

Solutions of these equations correspond to meromorphic functions on  $\mathbf{P}(\mathbb{V}^+)$ . The fact that  $\lambda$  itself is not a solution of these equation is a consequence of the fact that generally  $\mathbf{P}(\mathbb{V}^+)$  does not fibre over  $\mathbf{P}_1$  (equivalently  $\lambda$  is not a meromorphic function on  $\mathbf{P}(\mathbb{V}^+)$ ). In the case of hyper-Kähler or hyper-complex structures, where the spin-bundle does fibre over  $\mathbf{P}_1$ , the  $\lambda$  derivatives are not present in the Lax pair [5,8]. In these cases, one can reconstruct the transition functions of the bundle from the solutions of the above equations [11].

Although, Eq. (2.3) describes the most general anti-self-dual conformal structures locally, there are various special cases of these equations:

- Letting  $A_i = \lambda\phi_i$ , we recover the class of Hermitian anti-self-dual spaces, which are conformal to scalar-flat  $\bar{\partial}$ -Kähler metrics [14].
- Letting  $A_4 + \lambda A_2 = A_1 - \lambda A_3 = 0$  defines hyper-complex structures in four dimensions [5].
- Letting  $A_i = \lambda\phi_i$ , and assuming the vector fields  $\mathbf{e}_i$  are divergence free with respect to some volume element defines a scalar-flat Kähler metric up to a known conformal factor (this is an extension of a result of Park [13]).
- Letting  $A_4 + \lambda A_2 = A_1 - \lambda A_3 = 0$  and assuming the vector fields  $\mathbf{e}_i$  are divergence free with respect to some volume element defines a hyper-Kähler metric up to a known conformal factor [8].

Similar results hold for complex anti-self-dual conformal structures and real conformal structures of signature  $(-, -, +, +)$  with suitable generalisations and modifications of the reality conditions.

### 3. Grassmann structures

Anti-self-dual conformal structures in four dimensions are a special case of a more general type of structure, a right-flat Grassmann structure. Recall that for integers  $p, q \geq 2$ , a  $(p, q)$  Grassmann structure (or paraconformal structure in the terminology of Bailey and

Eastwood [2]) consists of a complex manifold  $M$  of complex dimension  $n = pq$  and an isomorphism  $\alpha$  between the (holomorphic) tangent bundle of  $M$  and the tensor product of a rank  $p$  complex vector bundle  $U$  with a rank  $q$  vector bundle  $V$

$$\alpha : TM \rightarrow U \otimes V. \tag{3.1}$$

Given such an isomorphism, we may introduce an isomorphism

$$\Lambda^p U \cong \Lambda^q V \tag{3.2}$$

between the highest exterior powers of these bundles.

Given connections, both denoted  $\nabla$ , on the bundles  $U$  and  $V$ , we may define a unique induced connection on  $TM$ , again denoted  $\nabla$ , by demanding that covariant differentiation commutes with the isomorphism  $\alpha$ . This affine connection naturally has torsion  $\mathbf{T}$  defined by

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = \mathbf{T}(\mathbf{X}, \mathbf{Y}) \quad \forall \mathbf{X}, \mathbf{Y} \in \Gamma(TM),$$

and curvature tensor  $\mathbf{R}$  given by

$$([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X}, \mathbf{Y}]})\mathbf{V} = \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{V} \quad \forall \mathbf{X}, \mathbf{Y}, \mathbf{V} \in \Gamma(TM).$$

A *scale* for a Grassmann structure consists of a non-vanishing section,  $\epsilon$ , of the bundle  $\Lambda^p U$ . The isomorphism (3.2) then implies the existence of a non-vanishing section,  $\epsilon'$ , of the bundle  $\Lambda^q V$ . It can be shown [2] that given the sections  $\epsilon$  and  $\epsilon'$ , there exist unique connections on  $U$  and  $V$  (and therefore on  $TM$ ) with the property that the torsion is trace-free, and which annihilate the forms  $\epsilon$  and  $\epsilon'$ :

$$\nabla\epsilon = \nabla\epsilon' = 0.$$

We shall generally assume the existence of a scale, and work with the unique connections which preserve it.

### 3.1. Algebraic decomposition of the torsion and curvature

If we consider the space of 2-forms on  $M$ , we have

$$\Lambda^2(T^*M) \cong \Lambda^2(U^* \otimes V^*) \cong (\Lambda^2(U^*) \otimes S^2(V^*)) \oplus (S^2(U^*) \otimes \Lambda^2(V^*)),$$

where  $\wedge$  and  $S$  denote skew-symmetric and symmetric powers of the relevant bundles, respectively. Viewing the torsion as a  $TM$  valued 2-form on  $M$ , it decomposes into two parts

$$\mathbf{T} = \mathbf{T}^+ \oplus \mathbf{T}^-,$$

where

$$\mathbf{T}^+ \in TM \otimes (\Lambda^2(U^*) \otimes S^2(V^*)), \quad \mathbf{T}^- \in TM \otimes (S^2(U^*) \otimes \Lambda^2(V^*)).$$

One can show that the trace-free parts of these parts of the torsion are independent of the connection chosen on the vector bundles  $U$  and  $V$  [2].

There is a similar decomposition of the curvature tensor  $\mathbf{R}$ . Given the direct product nature of  $TM$ , the curvature decomposes as a direct sum

$$\mathbf{R} = \mathbf{R}^- \otimes \text{Id}_V + \text{Id}_U \otimes \mathbf{R}^+,$$

where  $\mathbf{R}^-$  and  $\mathbf{R}^+$  denote the curvatures of the connections  $\nabla$  on  $U$  and  $V$ , respectively.  $\mathbf{R}^+$  may be viewed as a section of  $\Lambda^2(M) \otimes \text{End}(V)$ , and

$$\begin{aligned} \Lambda^2(M) \otimes \text{End}(V) &\cong ((\Lambda^2 U^* \otimes S^2 V^*) \oplus (S^2 U^* \otimes \Lambda^2 V^*)) \otimes (V \otimes V^*) \\ &\cong (\Lambda^2 U^* \otimes (V \otimes V^* \otimes S^2 V^*)) \oplus (S^2 U^* \otimes (V \otimes V^* \otimes \Lambda^2 V^*)). \end{aligned}$$

We now wish to consider the component of  $\mathbf{R}^+$  which is a section of  $\Lambda^2 U^* \otimes (V \otimes V^* \otimes S^2 V^*)$ . If we completely symmetrise in the  $V$  components, then the trace-free part of the remaining object will be referred to as the positive part of the Weyl tensor:

$$W^+ \in \Gamma(\text{trace-free part of } \Lambda^2 U^* \otimes (V^* \otimes S^3 V^*)).$$

**Definition 3.1.** A  $(p, q)$  Grassmann structure is right-flat if

$$T^+ = 0, \quad p > 2, \quad W^+ = 0, \quad p = 2.$$

In the case  $p > 2$ , the vanishing of the torsion implies automatically that  $W^+ = 0$ , whereas if  $p = 2$ , the torsion  $T^+$  automatically vanishes, so the condition  $W^+ = 0$  is non-trivial [2]. A complex four-dimensional spin-manifold with a metric is a particular case of a Grassmann manifold with  $p = q = 2$  since, due to the structure of the complexified rotation group, the complexified tangent bundle decomposes as a product of spin-bundles. In this case,  $W^+$  may be identified with the self-dual part of the Weyl tensor of the conformal structure [1]. In higher dimensions, with  $p = 2k$  and  $q = 2$ , special cases of right-flat Grassmann structures include quaternion-Kähler and hyper-Kähler structures.

### 3.2. Twistor spaces

In the case of right-flat Grassmann structures, there is an associated complex manifold  $Z$  of dimension  $(p + 1)(q - 1)$  which defines the structure. This manifold is constructed as follows.

Consider a  $(p, q)$  Grassmann structure on a complex manifold  $M$  as above, and assume we have a local basis  $\{\epsilon^a | a = 1, \dots, n\}$  for  $T^*M$ . The isomorphism (3.1) implies we may write this as  $\{\epsilon^{AA'} | A = 1, \dots, p, A' = 1, \dots, q\}$ . Given any  $\pi_x \in V_x$ , we define the annihilator

$$\pi_x^\perp := \{\phi_x \in V_x^* | \langle \phi_x, \pi_x \rangle = 0\} \subset V_x^*.$$

Let  $\Lambda \subset \Omega(V)$  be the distribution on the total space of the bundle  $V$  generated by the 1-forms

$$\sigma^A := \phi_{A'} \epsilon^{AA'}, \quad \phi \in \Gamma(\pi^\perp), \tag{3.3}$$

$$\sigma^{A'} := d\pi^{A'} + \gamma^{A'}_{B'} \pi^{B'}, \tag{3.4}$$

where  $\gamma^{A'}_{B'}$  denote the components of the connection on  $V$ . Complex conjugation gives the complex conjugate distribution  $\bar{\Lambda}$ . The sub-bundle of  $T(V)$  annihilated by  $\Lambda$  and  $\bar{\Lambda}$  is spanned by the distributions  $T$  and  $\bar{T}$ , where  $T \subset T(V)$  is spanned by the vector fields

$$v_A := \pi^{A'} D_{AA'}, \quad A = 1, \dots, p. \tag{3.5}$$

The distribution  $\Lambda$  is closed ( $d\Lambda \subset \Lambda^1 \wedge \Lambda$ ) if and only if the distribution  $T$  is closed under Lie brackets  $[T, T] \subset T$ . It is straightforward to show from Eq. (3.5) that given  $\lambda, \chi \in \Gamma(U)$

$$[\mathbf{v}_{\lambda \otimes \pi}, \mathbf{v}_{\chi \otimes \pi}] \equiv -\mathbf{T}(\lambda \otimes \pi, \chi \otimes \pi) + (\mathbf{R}(\lambda \otimes \pi, \chi \otimes \pi)\pi)^{A'} \frac{\partial}{\partial \pi^{A'}} \pmod{T}. \tag{3.6}$$

Therefore, if we wish the space  $T$  to be closed under Lie bracket, we require

$$\langle \phi, \mathbf{T}(\lambda \otimes \pi, \chi \otimes \pi) \rangle = 0 \quad \forall \phi \in \Gamma(\pi^\perp) \quad \forall \lambda, \chi \in \Gamma(U), \tag{3.7}$$

$$\mathbf{R}(\lambda \otimes \pi, \chi \otimes \pi)\pi = 0 \quad \forall \lambda, \chi \in \Gamma(U). \tag{3.8}$$

It is straightforward to show that if we fix the connections  $\nabla$  on  $U$  and  $V$  so as to preserve the scale, as mentioned in Section 3, then Eq. (3.8) implies that the Grassmann structure is right-flat. Therefore, the distribution  $T$  is integrable (equivalently  $\Lambda$  is closed under exterior differentiation) if and only if the Grassmann structure is right-flat.

We wish to consider the projective version of this construction. Treating the section  $\pi$  as homogeneous coordinates on the projective space  $P_{q-1} \cong \mathbf{P}(V)_p$  for each  $p \in M$ , we may introduce complex coordinates on the region  $U_1 = \{\pi \in \mathbb{C}^q \mid \pi^1 \neq 0\}$

$$\lambda^i = \frac{\pi^i}{\pi^1}, \quad i = 2, \dots, q.$$

The projections of the 1-forms above are

$$\sigma^A := \epsilon^{A1} + \lambda^2 \epsilon^{A2} + \dots + \lambda^q \epsilon^{Aq}, \quad A = 1, \dots, p, \tag{3.9}$$

$$\sigma^i := d\lambda^i - A^i, \quad i = 2, \dots, q, \tag{3.10}$$

where  $\mathbf{A}$  is the projective version of the connection. We again denote the distribution in  $\mathbf{P}(V)$  defined by these 1-forms by  $\Lambda$ . Similarly, a distribution, again denoted  $T \subset T(\mathbf{P}(V))$ , is spanned by the projection of the vector fields (3.5)

$$\mathbf{v}_A := D_1 + \lambda^2 D_2 + \dots + \lambda^q D_q, \quad A = 1, \dots, p. \tag{3.11}$$

The distribution spanned by these vector fields is integrable if and only if the Grassmann structure on  $M$  is right-flat. The integrability of this distribution implies we have a set of integrable  $p$ -dimensional planes in  $\mathbf{P}(V)$ . Quotienting out  $\mathbf{P}(V^+)$  by this distribution<sup>1</sup> therefore defines a quotient manifold  $Z$  of dimension  $(p + 1)(q - 1)$ , which we denote by

<sup>1</sup> We are assuming that there is nothing globally pathological about the fibration, and that such a quotient operation is justified.



$Z$ . We therefore have a map  $p : Z \rightarrow M$ , where the image of a point in  $Z$  is a  $p$ -dimensional plane in  $\mathbf{P}(V)$  (in twistorial terminology, an  $\alpha$ -plane). We can then define the distribution  $p^* \Lambda \subset \Lambda(Z)$ , which is involutive on  $Z$ . Since the dimension of  $p^* \Lambda$  equals the dimension of  $Z$ , this distribution therefore determines an almost-complex structure on  $Z$ . Moreover, since  $p^* \Lambda$  is involutive, this almost-complex structure is integrable. Therefore  $Z$  is a complex manifold of dimension  $(p + 1)(q - 1)$ .

We now wish to invert this process and construct the manifold  $M$  from a generic complex manifold  $Z$ . Given a point  $p \in M$ , its image in the manifold  $Z$  constructed above is a copy of  $P_{q-1} \subset Z$ , corresponding to the fibre  $\mathbf{P}(V)_p$ . We therefore wish to reconstruct the manifold  $M$  as the parameter space of embedded  $P_{q-1}$ 's in  $Z$ . In order to carry out this construction, we need to determine the normal bundle,  $N$ , of such an embedded sub-manifold.

In the notation of Eq. (3.4), the co-normal bundle,  $N^*$ , is spanned by the forms  $\{\phi_A \epsilon^{AA'} | \phi \in \pi^\perp \subset V^*, A = 1, \dots, p\}$ . The co-normal bundle is therefore isomorphic to  $p$  copies of the bundle  $\pi^\perp \subset V^*$ , which annihilates the element  $\pi \in V$ . Given  $x \in P_{q-1}$ ,  $\pi_x$  is an element of the complex line in  $C^q$  corresponding to the point  $x$ , i.e. an element of the  $L_x$ , where  $L$  denotes the tautological bundle  $L := H^{-1}$ . We define the Universal Quotient bundle,  $Q$ , so that the short sequence of vector bundles

$$0 \rightarrow L \rightarrow C^q \rightarrow Q \rightarrow 0$$

is exact, where  $C^q$  denotes the trivial rank  $q$  vector bundle over  $P_{q-1}$ . The bundle  $\pi^\perp$  is therefore isomorphic to  $Q^*$ , the dual of the quotient bundle. From the fact that  $Q \cong H \otimes TP_{q-1}$  [6], we deduce that

$$N^* \cong \bigoplus_1^p \Omega^1(1), \tag{3.12}$$

where for a general manifold  $X$ ,  $\Omega^r$  denotes the bundle of  $r$ -forms on  $X$ , and in the particular case of  $X = P_{q-1}$ , we define

$$\Omega^r(k) := \Omega^r(P_n) \otimes H^k.$$

The dual of Eq. (3.12) provides the normal bundle of  $P_{q-1} \subset Z$

$$N \cong \bigoplus_1^p H^{-1} \otimes T(P_{q-1}). \tag{3.13}$$

The Grassmann manifold  $M$  is reconstructed as the set of embedded  $P_{q-1}$ 's in  $Z$ . Given that we know the form of the normal bundle of an embedded  $P_{q-1}$  corresponding to a point  $x \in M$ , the number of deformations of the projective space follows from Kodaira's theorem: if  $H^1(P_{q-1}, N) = 0$ , then the space of embedded  $P_{q-1}$ 's is a complex analytic manifold  $M$ , and the tangent space,  $T_x M$ , is isomorphic to  $H^0(P_{q-1}, N)$ . To calculate these cohomology groups, we need some results concerning vector bundles over complex projective spaces [12]. Serre duality states that for a holomorphic vector bundle  $E$  over a (projective algebraic) complex  $n$  manifold  $X$ , we have the isomorphism

$$H^q(X, E) \cong (H^{n-q}(X, K_X \otimes E^*))^*,$$

where  $K_X$  denotes the canonical bundle of  $X$ . On such a manifold we also have the identification

$$(\Omega^r)^* \cong (\Omega^n)^* \otimes \Omega^{n-r}.$$

For a complex projective space

$$K_{P_n} \cong H^{-(n+1)},$$

so in this case, we have

$$H^q(P_n, \Omega^p(k)) \cong (H^{n-q}(P_n, \Omega^{n-p}(-k)))^*. \tag{3.14}$$

Results of Bott [3] then tell us that

$$\begin{aligned} \dim_{\mathbb{C}} H^q(P_n, \Omega^p(k)) &= \begin{cases} \binom{n+k-p}{k} \binom{k-1}{p}, & q=0, \quad 0 \leq p \leq n, \quad k > p, \\ 1, & k=0, \quad 0 \leq p=q \leq n, \\ \binom{-k+p}{-k} \binom{-k-1}{n-p}, & q=n, \quad 0 \leq p \leq n, \quad k < p-n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.15}$$

Applying these results, we first show that  $H^1(P_{q-1}, N) = 0$ . From Eqs. (3.12) and (3.14), we find that

$$\begin{aligned} H^1(P_{q-1}, N) &\cong (H^{q-2}(P_{q-1}, K \otimes N^*))^* \cong (H^{q-2}(P_{q-1}, H^{-q} \otimes \bigoplus_1^p \Omega^1(1)))^* \\ &\cong \bigoplus_1^p (H^{q-2}(P_{q-1}, \Omega^1(1-q)))^* \cong \bigoplus_1^p H^1(P_{q-1}, \Omega^{q-2}(q-1)) \cong 0, \end{aligned}$$

where the last equality follows from Eq. (3.15). Therefore  $T_x M$  is isomorphic to  $H^0(P_{q-1}, N)$ , where

$$H^0(P_{q-1}, N) \cong \bigoplus_1^p H^0(P_{q-1}, \Omega^{q-2}(q-1)) \cong \mathbb{C}^{pq},$$

by a similar argument to that given above. Therefore, given an embedded  $P_{q-1}$  in a complex manifold  $Z$  of complex dimension  $(p+1)(q-1)$  with normal bundle as in Eq. (3.13), there will exist an  $n = pq$  parameter family of such spaces. In the usual fashion, the integrability of the complex structure on  $Z$  then implies that  $M$  carries a right-flat Grassmann structure.

### 4. Integrable systems interpretation

The integrability of the distribution  $T$  defined by the vector fields (3.11) implies the existence of functions  $C_{AB}^C$  with the property that

$$[\mathbf{v}_A, \mathbf{v}_B] = \sum_{C=1}^p C_{AB}^C \mathbf{v}_C, \quad A, B = 1, \dots, p, \tag{4.1}$$

where, we recall, the vector fields  $\mathbf{v}_A$  are defined by

$$\mathbf{v}_A = D_1 + \lambda^2 D_2 + \dots + \lambda^q D_q$$

with

$$D_i = \mathbf{e}_i - A_i^j \frac{\partial}{\partial \lambda^j}$$

with the  $A_i^j$  quadratic polynomials in the spectral parameters  $\lambda^i$ . As such, the  $p$  vector fields  $\mathbf{v}_A$  are sections of the tangent bundle of  $P(V)$ , and correspond to differential operators which depend on a set of  $(q-1)$  spectral parameters  $(\lambda^2, \dots, \lambda^p)$ . More properly, these parameters correspond to a section of the line bundle  $H$  over  $P_{q-1}$ , so our “spectral parameter” now lives in  $P_{q-1}$ , unlike the usual case where we have a single spectral parameter in  $P_1$ . A power counting argument implies that the functions  $C_{AB}^C$  are quadratic polynomials in the complex coordinates  $\lambda^i$ , corresponding to sections of the bundle  $H^2$ .

As in the description of anti-self-dual conformal structures in four dimensions, the differential operators  $\mathbf{v}_A$  contain partial derivatives with respect to these spectral parameters, corresponding to the fact that the complex manifold  $Z$  generally does not fibre holomorphically over  $P_{q-1}$ .

The integrability of the distribution  $T$  is equivalent to the fact that the distribution  $\Lambda$  is involutive. Integrability of  $T$  implies the integrability of a distribution of  $p$ -dimensional planes in  $P(V)$ , and the existence of  $(p+1)(q-1)$  functions  $f^\alpha$  such that the planes are level sets of these functions. Equivalently, the distribution  $\Lambda$  is spanned by the differentials  $\{df^\alpha\}$ . If we then quotient out by the  $p$ -dimensional distribution to construct the manifold  $Z$ , then the functions  $f^\alpha$  descend to holomorphic functions on the manifold  $Z$ , and  $\{df^\alpha\}$  generate  $\Lambda^{(1,0)}(Z)$ .

In terms of the Grassmann manifold  $M$ , the  $\frac{1}{2}p(p-1)$  equations (4.1) are the integrability condition for over-determined set of equations for a function  $f(x : \lambda)$ :

$$\mathbf{v}_A f = 0, \quad A = 1, \dots, p. \tag{4.2}$$

When Eq. (4.1) are satisfied, there exist  $(p+1)(q-1)$  linearly independent solutions of these equations  $\{f_\alpha\}$ . The sub-space  $\{f_\alpha = \text{constant}\} \subset P(V)$  are then the  $\alpha$  planes of our right-flat Grassmann structure. The functions  $\{f_\alpha\}$  then descend to holomorphic functions on the quotient manifold  $Z$ .

In integrable systems terminology, Eq. (4.2) is the associated linear problem for the right-flat Grassmann structure. The compatibility condition Eq. (4.1) then ensures the integrability of the system. There are several non-standard elements of this construction, however. Firstly, the analogue of the spectral parameter of standard integrable systems theory in these equations is the set of affine coordinates  $\{\lambda^i\}$  on the  $(q-1)$ -dimensional complex projective space  $P_{q-1}$ .

Secondly, as opposed to the usual “Lax pair” formulation of integrable systems, we are here forced to consider a “Lax  $p$ -tuple” of operators, i.e. the vector fields  $\mathbf{v}_A$ , which must define an integrable distribution for the complex structure on the manifold  $Z$  to be integrable.

As in the simpler case of anti-self-dual conformal structures in dimension 4 (and similarly three-dimensional Einstein–Weyl structures), the differential operators we consider

generally contain derivatives in the spectral projective space, corresponding to the fact that the complex manifold  $Z$  generally does not fibre over the complex projective space  $P_{q-1}$ .

#### 4.1. Relations with Ward’s systems

Eq. (4.1) are, in some sense, an analogue of a construction due to Ward [16] for gauge fields. Ward considered principal  $G$ -bundles with a connection  $\mathbf{A} \in \Gamma(\Lambda^1 \otimes \mathfrak{g})$ . We consider a field  $\psi$  in a representation of  $G$ , and consider the over-determined set of linear equations

$$D_{\mathbf{V}_\alpha} \psi = 0, \quad \alpha = 1, \dots, p, \tag{4.3}$$

where  $D\psi$  denotes the covariant derivative of the field  $\psi$  with respect to the connection  $\mathbf{A}$ , and the  $\mathbf{V}_\alpha$  are vector fields. Moreover, the vector fields  $\mathbf{V}_\alpha$  are taken to depend on a set of complex parameters  $\{\lambda^A | A = 1, \dots, q\}$ , being a homogeneous polynomial of degree  $N$  in these parameters. (The vector fields may therefore be identified with a section of  $TM \otimes H^N$ , where  $H$  denotes the Hopf bundle over the complex projective space  $P_{q-1}$ .) For fixed  $\lambda^A$ , Eq. (4.3) are actually  $p \dim \mathfrak{g}$  differential equations for  $\dim \mathfrak{g}$  unknowns, and so are over-determined if  $p > 1$ . Since the system is over-determined, the existence of a maximal family of solutions places a set of algebraic constraints on the curvature  $\mathbf{F}$  of the connection

$$\mathbf{F}(\mathbf{V}_\alpha, \mathbf{V}_\beta) = 0, \quad \alpha, \beta = 1, \dots, p. \tag{4.4}$$

In the cases where the set of polynomial vector fields  $\mathbf{V}_\alpha$  are suitably non-degenerate, Eq. (4.4) can be completely solved by twistorial techniques [16].

The connection with Grassmann structures arises if we consider the case of linear polynomials corresponding to  $N = 1$ . In this case, the non-degeneracy condition mentioned above is analogous to the defining isomorphism (3.1). If we assume the underlying manifold of the theory is  $\mathbb{R}^{pq}$  with coordinates  $\{x^a : a = 1, \dots, pq\}$ , and that the connection is constant (i.e. independent of the  $x^a$ ), then the integrability conditions above become a set of algebraic equations on the connection

$$[\mathbf{A}(\mathbf{V}_\alpha), \mathbf{A}(\mathbf{V}_\beta)] = 0, \quad \alpha, \beta = 1, \dots, p.$$

If we now take the connection  $\mathbf{A}$  to have values in the tangent bundle of an auxiliary manifold  $\bar{M}$ , then we may write

$$\mathbf{A}(\mathbf{V}_\alpha) = \sum_{a=1}^n \sum_{A=1}^q \sigma_{\alpha A}^a \lambda^A \mathbf{e}_a, \tag{4.5}$$

where  $\{\mathbf{e}_a | a = 1, \dots, n\}$  denotes a basis of vector fields on the manifold  $\bar{M}$ . The integrability conditions (4.4) then reduce to a set of relations on the commutators of the vector fields  $\{\mathbf{e}_a\}$  on  $\bar{M}$

$$\sum_{A,B=0}^q \sum_{a,b=1}^n \lambda^A \lambda^B \sigma_{\alpha A}^a \sigma_{\beta B}^b [\mathbf{e}_a, \mathbf{e}_b] = 0, \tag{4.6}$$

where  $[, ]$  denotes the Lie bracket of vector fields. Imposing Eq. (4.6) for all values of the parameters  $\lambda^A$ , we recover Eq. (4.1) with  $C_{AB}{}^C = 0$  in the case when the derivatives with respect to the spectral parameter are not present.

If we allow derivatives with respect to the spectral parameter, then, in the terminology of Park [13], our equations for general right-flat Grassmann structures may therefore be considered as a  $P_{q-1}$ -extension of Ward’s equations for a constant connection on flat space with values in the tangent bundle of an auxiliary  $n$ -manifold  $\bar{M}$ .

A special case of these equations without derivatives with respect to the spectral parameters is Joyce’s interpretation of the equations for hyper-complex conformal structures in four dimensions [7], which in turn is a generalisation of the description of anti-self-dual Ricci-flat structures due to Mason and Newman [8].

In terms of Ward’s classification of systems in dimensions up to 11, Grassmann structures of type  $p = k, q = 2$  correspond to Ward’s systems  $A_k$ ,  $p = 2, q = m + 1$  correspond to his  $C_m$ , and  $p = q = 3$  correspond to his system  $D$ . The geometrical analogue of Ward’s systems with higher order homogeneous polynomials correspond to twistor spaces  $Z$  containing embedded  $P_{q-1}$ ’s with more complicated normal bundle, sections of which can be identified with a collection of sections of the bundle  $H^n$ . Unfortunately, there does not seem to be any simple geometrical interpretation of these systems in general.

### 5. Remarks and conclusions

If we wish to take seriously the idea that at the heart of classical integrable systems is a connection with complex geometry, then implicit in the formulation of Grassmann structures given in Eq. (4.1) are several generalisations of standard notions of integrability.

Firstly, the analogue of the spectral parameter of standard integrable systems theory in these equations is the set of affine coordinates  $\{\lambda^i\}$  on the  $(q - 1)$ -dimensional complex projective space  $P_{q-1}$ . In other words, the spectral parameter lives in a general complex projective space.

Secondly, as opposed to the usual “Lax pair” formulation of integrable systems, we are here forced to consider a “Lax  $p$ -tuple” of operators, i.e. the vector fields  $\mathbf{v}_A$ , which must define an integrable distribution for the complex structure on the manifold  $Z$  to be integrable.

Finally, as in the simpler case of anti-self-dual conformal structures in dimension 4 (and similarly three-dimensional Einstein–Weyl structures), the differential operators we consider generally contain derivatives in the spectral projective space, corresponding to the fact that the complex manifold  $Z$  generally does not fibre over the complex projective space  $P_{q-1}$ .

The only case in which we recover a standard Lax pair construction with spectral parameter in  $P_1$  is the case  $(p, q) = (2, 2)$ , when the twistor space fibres over  $P_1$ . Geometrically, this corresponds to the description of (complexified) hyper-complex structures in four dimensions.

The second observation above is consistent with the complex-manifold approach to hyper-complex and quaternionic–Kähler manifolds of real dimension  $4k$ , where the points of the manifold correspond to rational curves ( $q = 2$ ) with normal bundle  $\oplus_1^{2k} H$  in a complex manifold  $Z$  of dimension  $2k + 1$ . One could further generalise this picture by considering

more general embedded complex sub-manifolds than  $P_{q-1}$ , with more complicated normal bundles (see, for example [10]). The geometrical structures induced on the space of such sub-manifolds is, however, rather unclear, and does not appear to have any straightforward interpretation in terms of integrable systems. Even if we restrict ourselves to embedded rational curves, Grothendieck's theorem implies that the most general normal bundle is of the form  $N \cong \bigoplus_{i=1}^n H^{m_i}$  for integers  $m_i$ , but the geometrical interpretation of the induced structure on the space of rational curves for general  $m_i$  is far from apparent.

Finally, we should note that we have considered only complex Grassmann structures. If we consider an analytic real Grassmann manifold, where the complexified tangent bundle splits as a tensor product, then we may complexify the manifold and use the complex construction of the twistor space given above. However, there does not seem to be any straightforward definition of the twistor space in the case of non-analytic real Grassmann manifolds. The hope would be that analyticity follows from existence of a right-flat Grassmann structure, in the same way that in four dimensions the existence of an anti-self-dual conformal structure implies the existence of a real analytic structure [1]. From the twistorial point of view, we require that the complex manifold  $Z$  admit a real structure (i.e. an anti-holomorphic involution),  $\sigma$ , and that there be a  $n$ -parameter family of real embedded  $P_{q-1}$ 's which are invariant under this map. These invariant  $P_{q-1}$ 's then correspond to points of the manifold  $M$ . Since the manifold  $M$  is then a real sub-manifold of a complex-analytic manifold, it then necessarily admits a real-analytic structure. The existence (or not) of fixed points of  $\sigma$  then allows us to attribute a signature to the induced Grassmann structure on  $M$ , with the fixed point set generically defining a real projective space, which determines the set of null planes at a given point in  $M$ . A real structure on  $Z$  with no fixed points would define the analogue of a Riemannian structure of Theorem 1.1.

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